THE COX RING OF A SPHERICAL EMBEDDING

GIULIANO GAGLIARDI

Abstract. Let G be a connected reductive group and G/H a spherical homogeneous space. We show that the ideal of relations between a natural set of generators of the Cox ring of a G-embedding of G/H can be obtained by homogenizing certain equations which depend only on the homogeneous space. Using this result, we describe some examples of spherical homogeneous spaces such that the Cox ring of any of their G-embeddings is defined by one equation.

Introduction

Throughout the paper, we work with algebraic varieties and algebraic groups over the field of complex numbers $\mathbb{C}$.

Let Y be a normal irreducible variety whose divisor class group $\mathrm{Cl}(Y)$ is finitely generated and $\Gamma(Y, \mathcal{O}_Y^*) = \mathbb{C}^*$. The *Cox ring* of Y is the $\mathrm{Cl}(Y)$-graded $\mathbb{C}$-algebra

$$\mathcal{R}(Y) := \bigoplus_{[D]\in \mathrm{Cl}(Y)} \Gamma(Y, \mathcal{O}_Y(D)),$$

with the multiplication defined by the canonical maps

$$\Gamma(Y, \mathcal{O}_Y(D_1)) \otimes \Gamma(Y, \mathcal{O}_Y(D_2)) \to \Gamma(Y, \mathcal{O}_Y(D_1 + D_2)).$$

Some accuracy is required in order for this multiplication to be well-defined (cf. [Hau08] or [ADHL10] for details).

If follows from the results of Cox (cf. [Cox95]) that $\mathcal{R}(Y)$ is a polynomial ring if and only if Y is a toric variety. Toric varieties can be considered as examples in the more general class of spherical varieties, which are quasihomogeneous with respect to the action of a connected reductive group G.

The aim of this paper is to obtain a description of the Cox ring for an arbitrary spherical G-variety Y. Some results in this direction have been obtained by Brion (cf. [Bri07]). Our main interest is the ideal of relations between a natural set of generators of $\mathcal{R}(Y)$.

We recall some standard notions from the theory of spherical varieties. A closed subgroup $H \subseteq G$ of a connected reductive group G is called *spherical* if a Borel subgroup $B \subseteq G$ has an open orbit in G/H. In this case, G/H is called a *spherical homogeneous space*. Similarly to the theory of toric varieties, any G-equivariant open embedding $G/H \hookrightarrow Y$ into a normal G-variety Y can be described by some combinatorial data suggested by Luna and Vust (cf. [LV83] and [Kno91]). As the Cox ring does not depend on G-orbits of codimension two or greater, we may assume that Y contains only non-open G-orbits of codimension one.

Let $\mathcal{M}$ be the weight lattice of B-eigenvectors in the function field $\mathbb{C}(G/H)$. We denote by $\mathcal{N} := \mathrm{Hom}(\mathcal{M}, \mathbb{Z})$ the dual lattice. If $\nu : \mathbb{C}(G/H) \to \mathbb{Q}$ is a discrete valuation, its restriction to B-eigenvectors induces a map $u : \mathcal{M} \to \mathbb{Q}$, which lies in the vector space $\mathcal{N}_\mathbb{Q} := \mathcal{N} \otimes_\mathbb{Z} \mathbb{Q}$. We denote the set of G-invariant discrete valuations on $\mathbb{C}(G/H)$ by $\mathcal{V}$. The above assignment is injective on $\mathcal{V}$ and therefore defines an inclusion $\mathcal{V} \subseteq \mathcal{N}_\mathbb{Q}$. The set $\mathcal{V}$ is a cone called the *valuation cone* of G/H. We denote the G-invariant prime divisors in Y by $Y_1, \ldots, Y_n$. To each Y_l $(1 \le l \le n)$ we assign the element $u_l \in \mathcal{N}$ corresponding to the discrete valuation on $\mathbb{C}(G/H)$ induced

by Y_l. It follows from the Luna-Vust theory that the embedding $G/H \hookrightarrow Y$ can be described combinatorially by the fan Σ in $\mathcal{N}_\mathbb{Q}$ consisting of the one-dimensional cones $\sigma_l := \mathbb{Q}_{\geq 0} u_l$, which lie in the valuation cone $\mathcal{V}$.

We now give a summary of the results. Assume $\Gamma(Y, \mathcal{O}_Y^*) = \mathbb{C}^*$ and let $\mathcal{D} := \{D_1, \dots, D_r\}$ be the set of B-invariant prime divisors in G/H. The crucial case is the case where the homogeneous space G/H is quasiaffine with trivial divisor class group. In this case, we can choose prime elements $f_1, \dots, f_r \in \Gamma(G/H, \mathcal{O}_{G/H})$ with $\mathbb{V}(f_i) = D_i$ and obtain irreducible G-modules $V_i := \langle G \cdot f_i \rangle \subseteq \Gamma(G/H, \mathcal{O}_{G/H})$. For each i we set $s_i := \dim V_i$, choose a basis $\{f_{ij}\}_{j=1}^{s_i} \subseteq G \cdot f_i$ of V_i with $f_{i1} = f_i$, and we define $V := V_1 \oplus \ldots \oplus V_r$. We let m be the rank of the finitely generated free abelian group $\Gamma(G/H, \mathcal{O}_{G/H}^*)/\mathbb{C}^*$ and choose representatives of a basis $\{g_k\}_{k=1}^m$. The B-weights of the f_i and the g_k freely generate lattices $\mathcal{M}_V$ and $\mathcal{M}_T$ respectively, and we obtain $\mathcal{M} = \mathcal{M}_V \oplus \mathcal{M}_T \cong \mathbb{Z}^{r+m}$. Finally, we define the torus $T := \operatorname{Spec}(\mathbb{C}[\mathcal{M}_T])$.

The next step is to define a locally closed embedding

$$G/H \hookrightarrow Z := V^* \times T \cong \mathbb{C}^{s_1 + \ldots + s_r} \times (\mathbb{C}^*)^m.$$

We have $\mathbb{C}[Z] = S(V) \otimes_\mathbb{C} \mathbb{C}[\mathcal{M}_T]$, where $S(V)$ denotes the symmetric algebra of V. The coordinate ring of V_i^* is the symmetric algebra $S(V_i)$, whose generators corresponding to the above basis we denote by S_{ij} $(1 \leq j \leq s_i)$, i.e. $\mathbb{C}[V_i^*] = S(V_i) = \mathbb{C}[S_{i1}, \dots, S_{is_i}]$. We denote the generators of the coordinate ring of T corresponding to the above basis of $\mathcal{M}_T$ by T_k $(1 \leq k \leq m)$, i.e. $\mathbb{C}[T] = \mathbb{C}[\mathcal{M}_T] = \mathbb{C}[T_1^{\pm 1}, \dots, T_m^{\pm 1}]$. The locally closed embedding $G/H \hookrightarrow Z$ is then given by the G-equivariant surjective map $\mathbb{C}[Z] \to \Gamma(G/H, \mathcal{O}_{G/H})$ sending $S_{ij} \mapsto f_{ij}$ and $T_k \mapsto g_k$. Its kernel is the prime ideal $\mathbb{I}(G/H)$. Considering the natural action of the torus $\operatorname{Spec}(\mathbb{C}[\mathcal{M}])$ on Z, we obtain a corresponding $\mathcal{M}$-grading on the coordinate ring $\mathbb{C}[Z]$. For $f \in \mathbb{C}[Z]$ and $\mu \in \mathcal{M}$ we denote the μ-homogeneous component of f by $f^{(\mu)}$.

In order to describe the relations of $\mathcal{R}(Y)$, we define a homogenization operation in two steps. The first step is the map $\alpha : \mathbb{C}[Z] \to \big(\mathbb{C}[Z]\big)[W_1, \dots, W_n]$ defined as follows. For each $f \in \mathbb{C}[Z]$ and $u \in \mathcal{N}$ we define

$$\operatorname{ord}_u(f) := \min_{\mu \in \mathcal{M}} \Big\{ \langle u, \mu \rangle ; f^{(\mu)} \neq 0 \Big\},$$

and set

$$f^\alpha := \frac{\sum_{\mu \in \mathcal{M}} \Big(f^{(\mu)} \prod_{l=1}^n W_l^{\langle u_l, \mu \rangle} \Big)}{\prod_{l=1}^n W_l^{\operatorname{ord}_{u_l}(f)}}.$$

The second step is the map $\beta : \big(\mathbb{C}[Z]\big)[W_1, \dots, W_n] \to S(V)[W_1, \dots, W_n]$ sending $T_k \mapsto 1$ for each $1 \leq k \leq m$. Finally, we define the map $h : \mathbb{C}[Z] \to S(V)[W_1, \dots, W_n]$ by composing the two steps, i.e. $h := \beta \circ \alpha$. Note that we write the application of the maps α, β, and h as exponents, for example we write f^h instead of $h(f)$. We can now state our main result.

Main Theorem. *We have*

$$\mathcal{R}(Y) \cong S(V)[W_1, \dots, W_n] \big/ \big(f^h ; f \in \mathbb{I}(G/H) \big),$$

with $\operatorname{Cl}(Y)$*-grading given by* $\deg(S_{ij}) = [D_i]$ *and* $\deg(W_l) = [Y_l]$. *If* G *is of simply connected type and* H *is connected,* $\mathcal{R}(Y)$ *is a factorial ring.*

In particular, if $\mathbb{I}(G/H) = (f)$ is a principal ideal generated by f, then the ideal of relations of $\mathcal{R}(Y)$ is generated by f^h.

In the special case of a toric variety Y, the Main Theorem reduces to the result of Cox that $\mathcal{R}(Y)$ is a polynomial ring with one variable per G-invariant prime divisor because the homogeneous space has no B-invariant prime divisors, i.e. $V = \langle 0 \rangle$, and $\mathbb{I}(G/H) = (0)$. In the special case of a horospherical variety Y, we will see

that $\mathcal{R}(Y) \cong \mathcal{R}(G/P)[W_1, \ldots, W_n]$, where $P := N_G(H)$, i.e. the Cox ring of Y is a polynomial ring over the Cox ring of G/P.

The paper is organized in four sections. In Section 1 we present some information about the valuation cone $\mathcal{V}$ and describe it using tropical algebraic geometry. The proof of the Main Theorem in the crucial case of a quasiaffine spherical homogeneous space with trivial divisor class group is given in Section 2. Then we explain in Section 3 how the results can be extended to arbitrary spherical homogeneous spaces. Finally, we illustrate our results by some explicit examples in Section 4.

1. The valuation cone $\mathcal{V}$

It was shown in [Bri90] and [Kno94] that the valuation cone $\mathcal{V}$ is a fundamental chamber for the action of a crystallographic reflection group $W_{G/H}$ on $\mathcal{N}_\mathbb{Q}$ called the *little Weyl group* (cf. [Tim11, Theorem 22.13]). In particular, the cone $\mathcal{V}$ is cosimplicial.

More explicitly, in the situation of the introduction, we have the isotypic decomposition into G-modules

$$\Gamma(G/H, \mathcal{O}_{G/H}) = \bigoplus_{\mu \in \mathcal{M}} V_\mu,$$

where every irreducible G-module occurs with multiplicity at most one. We denote by Ψ the set of primitive elements $\mu \in \mathcal{M}$ such that $\{\langle \cdot, \mu \rangle = 0\}$ is a wall of $\mathcal{V}$ and $\langle \mathcal{V}, \mu \rangle \leq 0$. Then Ψ is a root system with Weyl group $W_{G/H}$. The elements of Ψ are called *spherical roots*. The cone $\mathcal{T}$ in $\mathcal{M} \otimes_\mathbb{Z} \mathbb{Q}$ generated by the spherical roots is the same as the closure of the cone generated by the elements of the form $\mu_1 + \mu_2 - \mu_3$ such that

$$V_{\mu_3} \subseteq V_{\mu_1} \cdot V_{\mu_2},$$

and $-\mathcal{V}$ is the dual cone of $\mathcal{T}$ (cf. [Los09, Section 3] and [Kno91, Section 5]).

In this sense, the valuation cone $\mathcal{V}$ is related to the failure of the $\mathbb{C}$-algebra $\Gamma(G/H, \mathcal{O}_{G/H})$ being $\mathcal{M}$-graded. We have the following extreme case. A spherical homogeneous space G/H is called *horospherical*, if $\mathcal{V} = \mathcal{N}_\mathbb{Q}$. In our setting, this is equivalent to the $\mathbb{C}$-algebra $\Gamma(G/H, \mathcal{O}_{G/H})$ being $\mathcal{M}$-graded. Another characterization of horospherical homogeneous spaces is that H contains a maximal unipotent subgroup of G (cf. [Pau84] and [Kno91, Corollary 6.2]).

We will state a result about the valuation cone which uses tropical algebraic geometry. An introduction to this subject can be found in [Mac12]. Let X be the toric variety associated to a fan Σ_X in $N_\mathbb{Q}$ with torus $\mathbb{T}$. If $S \subset \mathbb{T}$ is a closed subset, the tropicalization $\operatorname{trop}(S) \subseteq N_\mathbb{Q}$ of S is the support of a polyhedral fan in $N_\mathbb{Q}$. It gives an answer to the question of which $\mathbb{T}$-orbits in X intersect the closure $\overline{S}$ of S inside X. By a result of Tevelev (cf. [Tev07]), the $\mathbb{T}$-orbit corresponding to $\sigma \in \Sigma_X$ intersects $\overline{S}$ if and only if $\operatorname{trop}(S)$ intersects $\operatorname{relint}(\sigma)$.

In order to be able to use this machinery, we have to explain how Z can be regarded as a toric variety. We denote by M the finitely generated free abelian group with basis

$$\{S_{ij}, T_k; 1 \leq i \leq r, 1 \leq j \leq s_i, 1 \leq k \leq m\},$$

which is isomorphic to $\mathbb{Z}^{s_1 + \cdots + s_r + m}$. We define the torus $\mathbb{T} := \operatorname{Spec}(\mathbb{C}[M])$ with character lattice M, denote the dual lattice by $N := \operatorname{Hom}(M, \mathbb{Z})$, and set $N_\mathbb{Q} := N \otimes_\mathbb{Z} \mathbb{Q}$. The surjective map $M \to \mathcal{M}$ sending S_{ij} to the B-weight of f_i and T_k to the B-weight of g_k induces an inclusion $\mathcal{N}_\mathbb{Q} \hookrightarrow N_\mathbb{Q}$. The action of $\mathbb{T}$ on Z makes Z a toric variety.

Theorem 1.1. *We have*

$$\mathcal{V} = \operatorname{trop}(G/H \cap \mathbb{T}) \cap \mathcal{N}_{\mathbb{Q}}.$$

The proof will be given at the end of Section 2. We can compare this result to the approach Luna and Vust suggested in [LV83] using formal curves (cf. [Tim11, Chapter 24]), which shows that every G-invariant discrete valuation ν can be obtained up to proportionality by choosing a $\mathbb{C}(\!(t)\!)$-valued point $x(t)$ of G/H and defining

$$\nu(f) := \operatorname{ord}(f(g \cdot x(t)))$$

where $g \in G$ is a general point depending on $f \in \mathbb{C}(G/H)$.

On the other hand, taking into account the fundamental theorem of tropical algebraic geometry (cf. [SS04, Theorem 2.1]), Theorem 1.1 implies that any G-invariant discrete valuation ν comes from a $\mathbb{C}\{\{t\}\}$-valued point $x(t)$ of G/H satisfying

$$\operatorname{ord}(f_{ij}(x(t))) = \operatorname{ord}(f_i(x(t)))$$

for every i and j and is uniquely determined by

$$\nu(f_i) = \operatorname{ord}(f_i(x(t))) \text{ and } \nu(g_k) = \operatorname{ord}(g_k(x(t)))$$

for every i and k.

2. Proof of the Main Theorem

We begin by fixing the notation and gathering some preliminary facts. Let G be a connected reductive group and $H \subseteq G$ a spherical subgroup such that the spherical homogeneous space G/H is quasiaffine with trivial divisor class group, i.e. the ring $\Gamma(G/H, \mathcal{O}_{G/H})$ is factorial. Replacing G with a finite cover, we may assume that G is of simply connected type, i.e. $G = G^{ss} \times C$ where G^{ss} is semisimple simply connected and C is a torus.

We fix a Borel subgroup $B \subseteq G$ with subgroup of unipotent elements $U \subseteq B$. We denote the set of B-invariant prime divisors in G/H by $\mathcal{D} := \{D_1, \ldots, D_r\}$ and choose prime elements $f_1, \ldots, f_r \in \Gamma(G/H, \mathcal{O}_{G/H})$ with $\mathbb{V}(f_i) = D_i$.

Remark 2.1. Each f_i is a B-eigenvector, and for each f_i the G-module spanned by $G \cdot f_i$ in $\Gamma(G/H, \mathcal{O}_{G/H})$ is irreducible and finite-dimensional (cf. [Kra84, III.1.5]). By [KKV89, Proposition 1.3], all elements of $\Gamma(G/H, \mathcal{O}^*_{G/H})$ are G-eigenvectors and the quotient $\Gamma(G/H, \mathcal{O}^*_{G/H})/\mathbb{C}^*$ is a finitely generated free abelian group.

For each $0 \leq i \leq r$ we set $V_i := \langle G \cdot f_i \rangle \subseteq \Gamma(G/H, \mathcal{O}_{G/H})$ and $s_i := \dim V_i$. We choose a basis $\{f_{ij}\}_{j=1}^{s_i} \subseteq G \cdot f_i$ of V_i with $f_{i1} = f_i$. Then the f_{ij} are pairwise nonassociated prime elements. We set $V := V_1 \oplus \ldots \oplus V_r$. We denote by m the rank of the free abelian group $\Gamma(G/H, \mathcal{O}^*_{G/H})/\mathbb{C}^*$ and choose representatives of a basis $\{g_k\}_{k=1}^m$.

Proposition 2.2. *The B-eigenvectors in $\Gamma(G/H, \mathcal{O}_{G/H})$ are given by*

$$\Gamma(G/H, \mathcal{O}_{G/H})^{(B)} = \left\{ c f_1^{d_1} \cdots f_r^{d_r}; c \in \Gamma(G/H, \mathcal{O}^*_{G/H}), d_i \in \mathbb{N}_0 \right\}.$$

Proof. For each $f \in \Gamma(G/H, \mathcal{O}_{G/H})^{(B)}$ all irreducible components of $\mathbb{V}(f)$ are B-invariant since B is irreducible. □

Proposition 2.3. *The $\mathbb{C}$-algebra $\Gamma(G/H, \mathcal{O}_{G/H})$ is generated by $\{f_{ij}, g_k^{\pm 1}\}$.*

Proof. Since every element of $\Gamma(G/H, \mathcal{O}_{G/H})^U$ can be written as the sum of B-eigenvectors, the $\mathbb{C}$-algebra $\Gamma(G/H, \mathcal{O}_{G/H})^U$ is generated by $\{f_i, g_k^{\pm 1}\}$. By [Kra84, III.3.1], it follows that $\Gamma(G/H, \mathcal{O}_{G/H})$ is generated as claimed. □

For each $f \in \mathbb{C}(G/H)^{(B)}$ we denote by χ_f its B-weight. We define the weight lattice of B-eigenvectors in $\mathbb{C}(G/H)$ by

$$\mathcal{M} := \{\chi \in \mathfrak{X}(B); \text{there exists } f \in \mathbb{C}(G/H)^{(B)} \text{ with } \chi = \chi_f\}.$$

We denote by $\mathcal{N} := \operatorname{Hom}(\mathcal{M}, \mathbb{Z})$ the dual lattice, set $\mathcal{N}_{\mathbb{Q}} := \mathcal{N} \otimes_{\mathbb{Z}} \mathbb{Q}$, and remember that there is an exact sequence (cf. [Kno91, after 1.7])

$$1 \to \mathbb{C}^* \to \mathbb{C}(G/H)^{(B)} \to \mathcal{M} \to 0.$$

We denote by $\mathcal{V} \subseteq \mathcal{N}_{\mathbb{Q}}$ the valuation cone of G/H (cf. [Kno91, after 1.8]). We also set $v_i^* := \chi_{f_i}$ and $w_k^* := \chi_{g_k}$.

Proposition 2.4. *The lattice $\mathcal{M} \subseteq \mathfrak{X}(B)$ is freely generated by*

$$\{v_1^*, \dots, v_r^*, w_1^*, \dots, w_m^*\}.$$

Proof. Every $f \in \mathbb{C}(G/H)^{(B)}$ can be written as $\frac{g}{h}$ with $g, h \in \Gamma(G/H, \mathcal{O}_{G/H})^{(B)}$ since $\operatorname{Supp}(\operatorname{div}(f))$ is the union of B-invariant prime divisors. The claim then follows from Proposition 2.2, the exact sequence above, and the fact that $\{g_k\}_{k=1}^m$ is a basis of $\Gamma(G/H, \mathcal{O}^*_{G/H})/\mathbb{C}^*$. □

We denote the corresponding dual basis of $\mathcal{N}$ by $\{v_1, \dots, v_r, w_1, \dots, w_m\}$. We also denote by $\mathcal{M}_V$ and $\mathcal{M}_T$ the sublattices of $\mathcal{M}$ generated by $\{v_1^*, \dots, v_r^*\}$ and $\{w_1^*, \dots, w_m^*\}$ respectively and obtain $\mathcal{M} = \mathcal{M}_V \oplus \mathcal{M}_T \cong \mathbb{Z}^{r+m}$. Finally, we define the torus $T := \operatorname{Spec}(\mathbb{C}[\mathcal{M}_T])$.

The next step is to define a G-equivariant locally closed embedding

$$G/H \hookrightarrow Z := V^* \times T \cong \mathbb{C}^{s_1 + \dots + s_r} \times (\mathbb{C}^*)^m.$$

We have $\mathbb{C}[Z] = S(V) \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{M}_T]$. The coordinate ring of V_i^* is the symmetric algebra $S(V_i)$, whose generators corresponding to the above basis we denote by S_{ij} $(1 \leq j \leq s_i)$, i.e. $\mathbb{C}[V_i^*] = S(V_i) = \mathbb{C}[S_{i1}, \dots, S_{is_i}]$. We denote the generators of the coordinate ring of T corresponding to the above basis of $\mathcal{M}_T$ by T_k $(1 \leq k \leq m)$, i.e. $\mathbb{C}[T] = \mathbb{C}[\mathcal{M}_T] = \mathbb{C}[T_1^{\pm 1}, \dots, T_m^{\pm 1}]$.

The locally closed embedding is then given by the map

$$\begin{aligned} \Phi : S(V) \otimes_{\mathbb{C}} \mathbb{C}[T] &\to \Gamma(G/H, \mathcal{O}_{G/H}) \\ S_{ij} &\mapsto f_{ij} \\ T_k &\mapsto g_k, \end{aligned}$$

which is surjective by Proposition 2.3. This map is G-equivariant, hence the locally closed embedding $G/H \hookrightarrow Z$ is G-equivariant with respect to the corresponding G-action on Z. We set $\mathfrak{p} := \mathbb{I}(G/H) = \ker \Phi$.

Remark 2.5. The action of G on

$$Z \cong \mathbb{C}^{s_1} \times \ldots \times \mathbb{C}^{s_r} \times \mathbb{C}^* \times \ldots \times \mathbb{C}^*$$

is linear on each factor.

The natural action of the torus $\operatorname{Spec}(\mathbb{C}[\mathcal{M}])$ on Z defines a corresponding $\mathcal{M}$-grading on $\mathbb{C}[Z]$.

Proposition 2.6. *The prime ideal $\mathfrak{p}$ is $\mathcal{M}$-graded, i.e. $\Gamma(G/H, \mathcal{O}_{G/H})$ is a $\mathcal{M}$-graded $\mathbb{C}$-algebra, if and only if G/H is horospherical.*

Proof. This follows from [Tim11, Proposition 7.6]. □

We denote by M the finitely generated free abelian group with basis

$$\{S_{ij}, T_k; 1 \leq i \leq r, 1 \leq j \leq s_i, 1 \leq k \leq m\},$$

which is isomorphic to $\mathbb{Z}^{s_1+\ldots+s_r+m}$. We define the torus $\mathbb{T} := \operatorname{Spec}(\mathbb{C}[M])$ with character lattice M, denote the dual lattice by $N := \operatorname{Hom}(M, \mathbb{Z})$, and set $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. The surjective map $M \to \mathcal{M}$ sending $S_{ij} \mapsto v_i^*$ and $T_k \mapsto w_k^*$ induces an inclusion $\mathcal{N}_{\mathbb{Q}} \hookrightarrow N_{\mathbb{Q}}$. The action of $\mathbb{T}$ on Z makes Z a toric variety.

Lemma 2.7. *Every non-open G-orbit of the spherical embedding*

$$G/H \hookrightarrow \overline{G/H} \subseteq Z$$

is contained in the closure of a B-invariant prime divisor in G/H.

Proof. As the ring $\Gamma(G/H, \mathcal{O}_{G/H})$ is factorial, the non-open G-orbits are at least of codimension two. If follows from the general theory of spherical embeddings (cf. [Kno91]) that all non-open G-orbits lie in the closure of a B-invariant prime divisor in G/H if there is no G-invariant prime divisor of codimension one. □

Proposition 2.8. *Let $Z_i := \mathbb{V}(S_{ij}; 1 \leq j \leq s_i) \subseteq Z$. Then we have*

$$G/H = \overline{G/H} \setminus (Z_1 \cup \ldots \cup Z_r).$$

Proof. Let A be a non-open G-orbit in $\overline{G/H}$. By Lemma 2.7, there exists a D_i with $A \subseteq \overline{D_i}$. It follows that S_{i1} vanishes on A. Since V_i is an irreducible G-module and A is G-invariant, the whole module vanishes on A, so $A \subseteq Z_i$ follows. □

We set

$$X_0 := Z \setminus (Z_1 \cup \ldots \cup Z_r).$$

Then $G/H \hookrightarrow X_0$ is a closed embedding, and G and $\mathbb{T}$ act on X_0.

We consider a spherical embedding of G/H which contains only non-open G-orbits of codimension one and is therefore in particular smooth. It is given by a fan Σ in $\mathcal{N}_{\mathbb{Q}}$ which consists of one-dimensional cones $\sigma_l = \mathbb{Q}_{\geq 0} u_l$ $(1 \leq l \leq n)$ where $u_l \in \mathcal{N} \cap \mathcal{V}$.

We now give an outline of the next steps. Consider the fan Σ_{X_0} in $N_{\mathbb{Q}}$ corresponding to the toric variety X_0. Our plan is to define a fan Σ_X extending the fan Σ_{X_0} such that the closure of G/H inside the toric variety X associated to the fan Σ_X is the spherical embedding corresponding to the fan Σ. We have already established the natural inclusion $\mathcal{N}_{\mathbb{Q}} \subseteq N_{\mathbb{Q}}$, so the cones of the fan Σ in $\mathcal{N}_{\mathbb{Q}}$ can as well be considered as cones in $N_{\mathbb{Q}}$. But extending Σ_{X_0} by these one-dimensional cones is not enough, since it might be impossible to extend the action of G on X_0 to the resulting toric variety X. The plan can be carried out, however, if we also add some higher-dimensional cones.

We construct the fan Σ_X in $N_{\mathbb{Q}}$ as follows. We first define the set

$$\mathfrak{A} := \{\mathfrak{a} \subseteq \{v_{ij}\}; \text{for each } i \text{ there is exactly one } j \text{ with } v_{ij} \notin \mathfrak{a}\}.$$

For each $1 \leq l \leq n$ and $\mathfrak{a} \in \mathfrak{A}$ we define the cone

$$\sigma_{l,\mathfrak{a}} := \operatorname{cone}(\{u_l\} \cup \mathfrak{a}) \subseteq N_{\mathbb{Q}}$$

and set

$$\Sigma_X := \operatorname{fan}(\{\sigma_{l,\mathfrak{a}}; 1 \leq l \leq n, \mathfrak{a} \in \mathfrak{A}\}),$$

the fan generated by the $\sigma_{l,\mathfrak{a}}$ in $N_{\mathbb{Q}}$. It is not difficult to see that the $\sigma_{l,\mathfrak{a}}$ are strictly convex and compatible, so Σ_X is well-defined. We denote by X the toric variety associated to the fan Σ_X, which is smooth. Note that Σ_X extends Σ_{X_0}, so we have an open embedding $X_0 \subseteq X$.

We will now show that the G-action on X_0 can be extended to X. We consider $\widehat{N} := N \oplus \mathbb{Z}^n$ and regard $N_{\mathbb{Q}} \subseteq \widehat{N}_{\mathbb{Q}} := \widehat{N} \otimes_{\mathbb{Z}} \mathbb{Q}$ as naturally included. We denote the standard basis of $\mathbb{Z}^n$ by $\{e_l\}_{l=1}^n$. We set

$$\widehat{\sigma}_{l,\mathfrak{a}} := \operatorname{cone}(\{e_l\} \cup \mathfrak{a}) \subseteq \widehat{N}_{\mathbb{Q}}$$

and

$$\Sigma_{\widehat{X}} := \operatorname{fan}(\{\widehat{\sigma}_{l,\mathfrak{a}}; 1 \le l \le n, \mathfrak{a} \in \mathfrak{A}\}).$$

The associated quasiaffine toric variety $\widehat{X}$ comes with a natural toric morphism

$$p : \widehat{X} \to X$$

defined by the lattice map $\widehat{N} \to N$ sending $v_{ij} \mapsto v_{ij}$, $w_k \mapsto w_k$, and $e_l \mapsto u_l$.

Remark 2.9. We have

$$\widehat{X} \cong V^* \times T \times \mathbb{A}^n \setminus \widehat{S},$$

where $\widehat{S}$ is a closed subset of codimension at least two. According to the theory in [Świ99], the toric morphism $p : \widehat{X} \to X$ is a good quotient for the action of a subtorus $\Gamma \subseteq \mathbb{T} \times (\mathbb{C}^*)^n$. It is even a geometric quotient. The subtorus Γ has a natural parametrization $\kappa : (\mathbb{C}^*)^n \xrightarrow{\cong} \Gamma$. Denoting by W_l ($1 \le l \le n$) the coordinates of $(\mathbb{C}^*)^n$, the action of Γ on $\widehat{X}$ is given by

$$\kappa(t) \cdot v = \left(\prod_{l=1}^n W_l(t)^{-\langle u_l, v_i^* \rangle} \right) \cdot v,$$

on $v \in V_i^*$, similarly, by

$$\kappa(t) \cdot v = \left(\prod_{l=1}^n W_l(t)^{-\langle u_l, w_k^* \rangle} \right) \cdot v,$$

on v in the k-th factor $\mathbb{C}^*$ of $T \cong (\mathbb{C}^*)^m$, and finally by the natural action on $\mathbb{A}^n$.

We let G act linearly on the first factors of $\widehat{X}$ with the same action as in Remark 2.5 and trivially on $\mathbb{A}^n$. Then the action of G on $\widehat{X}$ commutes with the action of the torus Γ from Remark 2.9. In particular, we obtain an action of G on X.

Remark 2.10. The natural inclusion $N_{\mathbb{Q}} \subseteq \widehat{N}_{\mathbb{Q}}$ defines a G-equivariant toric closed embedding

$$\psi : X_0 \to \widehat{X},$$

and we obtain the following commutative diagram.

$$\begin{tikzcd} & \widehat{X} \arrow[dr, "p"] & \\ X_0 \arrow[ur, "\psi"] \arrow[rr, hook] & & X \end{tikzcd}$$

This shows that the action of G on X is an extension of the action of G on X_0.

We denote by Y the closure of G/H inside X. It will now take some time to prove that Y is indeed the spherical embedding of G/H associated to the fan Σ.

Proposition 2.11. *The preimage $p^{-1}(G/H)$ with the action of $G \times \Gamma$ is a spherical homogeneous space isomorphic to $G/H \times \Gamma$ with the natural action of $G \times \Gamma$. The isomorphism*

$$G/H \times \Gamma \xrightarrow{\cong} p^{-1}(G/H)$$

is given by $(x,t) \mapsto t \cdot \psi(x)$.

Proof. As p is a geometric quotient, it follows that $p^{-1}(G/H) = \Gamma \cdot \psi(G/H)$. $\square$

Remark 2.12. If we denote by W_l ($1 \le l \le n$) the coordinates of $\mathbb{A}^n$ as well as the coordinates of Γ under the parametrization κ from Remark 2.9, we have $\mathbb{C}[\mathbb{A}^n] = \mathbb{C}[W_1, \dots, W_n]$ and $\mathbb{C}[\Gamma] = \mathbb{C}[W_1^{\pm 1}, \dots, W_n^{\pm 1}]$. Considering the natural inclusion

$$p^{-1}(G/H) \hookrightarrow V^* \times T \times \mathbb{A}^n,$$

the isomorphism of Proposition 2.11 is induced by the map

$$\begin{aligned} S(V) \otimes_{\mathbb{C}} \mathbb{C}[T] \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{A}^n] &\to (S(V) \otimes_{\mathbb{C}} \mathbb{C}[T])/\mathfrak{p} \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \\ S_{ij} &\mapsto S_{ij} \otimes \left(\prod_{l=1}^{n} W_l^{-\langle u_l, v_i^* \rangle} \right) \\ T_k &\mapsto T_k \otimes \left(\prod_{l=1}^{n} W_l^{-\langle u_l, w_k^* \rangle} \right) \\ W_l &\mapsto W_l. \end{aligned}$$

We denote the closure of $p^{-1}(G/H)$ inside $\widehat{X}$ by $\widehat{Y}$.

Proposition 2.13. $\widehat{Y}$ *is the quasiaffine spherical embedding of* $G/H \times \Gamma$ *corresponding to the fan* $\widehat{\Sigma}$ *in* $(\mathcal{N} \oplus \mathbb{Z}^n)_{\mathbb{Q}}$ *which consists of the one-dimensional cones*

$$\mathbb{Q}_{\ge 0}(u_l + e_l)$$

for $1 \le l \le n$. *Furthermore, the ring* $\Gamma(\widehat{Y}, \mathcal{O}_{\widehat{Y}})$ *is factorial.*

Proof. Consider the spherical embedding associated to the fan $\widehat{\Sigma}$, which is quasiaffine with factorial ring of global sections (cf. [Bri07, Proposition 4.1.1]). By possibly adding some orbits of codimension at least two, we can make it affine (cf. [Kno91, Proof of Theorem 6.7]) and denote its coordinate ring by R. We will show that it is the closure of $p^{-1}(G/H)$ inside the affine toric variety $V^* \times T \times \mathbb{A}^n$. Consider the following diagram.

$$\begin{tikzcd} S(V) \otimes_{\mathbb{C}} \mathbb{C}[T] \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{A}^n] \arrow[r] & R \arrow[d, hook] & \\ & (S(V) \otimes_{\mathbb{C}} \mathbb{C}[T])/\mathfrak{p} \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \arrow[r, equal] & \Gamma(G/H \times \Gamma, \mathcal{O}_{G/H \times \Gamma}) \end{tikzcd}$$

As the ring R is normal, it consists of all the elements in the ring of global sections of $G/H \times \Gamma$ which have a nonnegative value under the valuations induced by the G-invariant prime divisors in $\operatorname{Spec}(R)$. It follows that the elements which are required to define the horizontal map as in Remark 2.12 belong to R.

Additionally, all the $B \times \Gamma$-eigenvectors in R, and even the $G \times \Gamma$-modules generated by them, belong to the image of the horizontal map, so we obtain surjectivity of the horizontal map with the same argument as in Proposition 2.3. The composition of the horizontal map and the vertical localization is the same map as in Proposition 2.11. It follows that we have $\overline{p^{-1}(G/H)} = \operatorname{Spec}(R)$ inside $V^* \times T \times \mathbb{A}^n$.

We use the same symbols for elements of $S(V) \otimes_{\mathbb{C}} \mathbb{C}[T] \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{A}^n]$ and their images in R. As the valuations induced by the G-invariant prime divisors in $\operatorname{Spec}(R)$ have values of 0 or 1 respectively on the W_l and of 0 on the S_{ij}, it follows that the W_l and the S_{ij} are pairwise nonassociated prime elements of R. The G-orbits which lie in the closure of $D_i \times \Gamma$ are contained in $\mathbb{V}(S_{ij}; 1 \le j \le s_i)$, while the other

G-orbits of codimension at least two are contained in $\mathbb{V}(W_{l_1}) \cap \mathbb{V}(W_{l_2})$ for some $l_1 \neq l_2$. Therefore intersecting with $\widehat{X}$, i.e. removing the set $\widehat{S}$, removes exactly the G-orbits of codimension at least two, and the result follows. $\square$

We will re-use the following fact from the proof of Proposition 2.13.

Remark 2.14. The W_l and the S_{ij} are pairwise nonassociated prime elements of $\Gamma(\widehat{Y}, \mathcal{O}_{\widehat{Y}})$.

Remark 2.15. The restriction

$$p|_{\widehat{Y}} : \widehat{Y} \to Y$$

is a good geometric quotient. In particular, Y is normal.

Lemma 2.16. *Assume that Σ contains only one one-dimensional cone σ which does not necessarily belong to $\mathcal{V}$. This means that Σ might not be well-defined, but we can still construct X and consider the closure of G/H inside X. Any irreducible component of the complement to G/H in the closure of G/H inside X which intersects the $\mathbb{T}$-orbit corresponding to σ induces the G-invariant valuation which defines σ (after possibly normalizing).*

Proof. We consider the affine open toric subvariety U_σ of X with two orbits, the torus and the orbit corresponding to the cone σ. Let $\pi \in \mathbb{C}[U_\sigma]$ be the prime element such that $\mathbb{V}(\pi)$ is the orbit corresponding to the cone σ. We obtain the following commutative diagram.

$$\begin{tikzcd}
\mathbb{C}[U_\sigma] \arrow[rrr, hook] \arrow[d] & & & (S(V) \otimes_{\mathbb{C}} \mathbb{C}[T])_{\mathfrak{p}} \arrow[d] & \\
\mathbb{C}[U_\sigma]/(\mathfrak{p} \cap \mathbb{C}[U_\sigma]) \arrow[r, hook, "\mathfrak{N}"] & R_1 \arrow[r, hook, "\mathfrak{L}"] & R_2 \arrow[r, hook] & (S(V) \otimes_{\mathbb{C}} \mathbb{C}[T])_{\mathfrak{p}}/\mathfrak{p} \arrow[r, equal] & \mathbb{C}(G/H)
\end{tikzcd}$$

It is sufficient to localize at the prime ideal $\mathfrak{p}$ to get the inclusion in the first row since the S_{ij} are not in $\mathfrak{p}$. In the bottom row, $\mathfrak{N}$ denotes normalization, and $\mathfrak{L}$ is localization in such a way that $\mathbb{V}(\pi) = \mathbb{V}(\pi_0)$ in $\operatorname{Spec}(R_2)$ where $\pi_0 \in R_2$ is a prime element. Each $L \in \{S_{ij}, T_k\}$ can be written as $L = c_1\pi^{d_1}/\pi^{d_2}$ with $c_1 \in \mathbb{C}[U_\sigma]^*$ and $d_1, d_2 \in \mathbb{N}_0$ since there are no $\mathbb{T}$-invariant prime divisors other than $\mathbb{V}(\pi)$ in U_σ. Therefore we have $L = c_2\pi_0^{d_3d_1}/\pi_0^{d_3d_2}$ for some $c_2 \in R_2^*$ and $d_3 \in \mathbb{N}_0$. It follows that $\mathbb{V}(\pi_0)$ defines the correct cone σ. $\square$

Proposition 2.17. *Y is the spherical embedding of G/H corresponding to the fan Σ.*

Proof. As general embeddings of G/H can be obtained by gluing embeddings with only one closed orbit, we may assume $n = 1$. In this case, it is clear that Y has two G-orbits, and Lemma 2.16 implies that the closed G-orbit induces the correct G-invariant valuation. $\square$

We denote by $Y_1, \dots, Y_n$ the G-invariant prime divisors in Y and by $X_1, \dots, X_n$ the $\mathbb{T}$-invariant prime divisors in X corresponding to $u_1, \dots, u_n$ respectively.

We now turn to the Cox ring of Y. From now on, we assume $\Gamma(Y, \mathcal{O}_Y^*) = \mathbb{C}^*$, which implies $\Gamma(X, \mathcal{O}_X^*) = \mathbb{C}^*$.

We temporarily re-use our notation in a more general setting to give an overview over the next step. Let X be a toric variety with $\Gamma(X, \mathcal{O}_X^*) = \mathbb{C}^*$ and $\mathbb{T}$-invariant prime divisors $X_1, \dots, X_n$. Then we have the canonical quotient construction $\pi : \widetilde{X} \to X$, where $\widetilde{X}$ is a quasiaffine toric variety and $\mathcal{R}(X) \cong \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ (cf. [CLS11, Theorem 5.1.11]). Let $\iota : Y \hookrightarrow X$ be a closed embedding with $\Gamma(Y, \mathcal{O}_Y^*) = \mathbb{C}^*$. It follows from the work of Hausen (cf. [Hau08]) that $\mathcal{R}(Y) \cong \Gamma(\pi^{-1}(Y), \mathcal{O}_{\pi^{-1}(Y)})$ if

the closed embedding satisfies certain conditions. In case Y and X are smooth, the conditions are that $X_l \cap Y$ is an irreducible hypersurface in Y intersecting the $\mathbb{T}$-orbit which is dense in X_l for each $1 \le l \le n$ and that the map $\iota^* : \mathrm{Cl}(X) \to \mathrm{Cl}(Y)$ induced by the pullback of Cartier divisors is an isomorphism. Such an embedding is called *neat*.

We now return to our setting. We identify the prime divisors D_i in G/H and $\mathbb{V}(S_{ij})$ in X_0 with their closures inside Y and X respectively as long as there is no danger of confusion.

Proposition 2.18. *The closed embedding*

$$\iota : Y \hookrightarrow X$$

is a neat embedding in the sense of [Hau08, Definition 2.5].

Proof. Consider the following commutative diagram.

$$\begin{tikzcd} \widehat{Y} \arrow[r, hook, "\widehat{\iota}"] \arrow[d, "p"'] & \widehat{X} \arrow[d, "p"] \\ Y \arrow[r, hook, "\iota"'] & X \end{tikzcd}$$

Since the diagram commutes, $\iota^{-1}(X_l)$ and $\iota^{-1}(\mathbb{V}(S_{ij}))$ are irreducible for each i, j, and l, and using Remark 2.14 as in the proof of Theorem 1.1, we obtain that they intersect the corresponding $\mathbb{T}$-orbit of codimension one in X.

We have a pullback map of Cartier divisors $\iota^* : \mathrm{Div}(X) \to \mathrm{Div}(Y)$, and clearly $\mathrm{Supp}(\iota^*(X_l)) \subseteq Y_l$ and $\mathrm{Supp}(\iota^*(\mathbb{V}(S_{i1}))) \subseteq D_i$. Using the diagram again, we see that locally the pullbacks of X_l and $\mathbb{V}(S_{i1})$ are prime divisors. Therefore we have $\iota^*(X_l) = Y_l$ and $\iota^*(\mathbb{V}(S_{i1})) = D_i$.

Using the explicit descriptions of the divisor class groups of toric and spherical varieties (cf. [CLS11, Theorem 4.1.3] and [Bri07, Proposition 4.1.1]), we obtain that the induced pullback map $\iota^* : \mathrm{Cl}(X) \to \mathrm{Cl}(Y)$ is an isomorphism. $\square$

We are now almost ready to describe the Cox ring $\mathcal{R}(Y)$. We first define a homogenization operation in two steps. The first step is the map $\alpha : \mathbb{C}[Z] \to \big(\mathbb{C}[Z]\big)[W_1, \ldots, W_n]$ defined as follows. For each $f \in \mathbb{C}[Z]$ and $u \in \mathcal{N}$ we define

$$\mathrm{ord}_u(f) := \min_{\mu \in \mathcal{M}} \Big\{ \langle u, \mu \rangle ; f^{(\mu)} \neq 0 \Big\},$$

and set

$$f^\alpha := \frac{\sum_{\mu \in \mathcal{M}} \Big(f^{(\mu)} \prod_{l=1}^n W_l^{\langle u_l, \mu \rangle} \Big)}{\prod_{l=1}^n W_l^{\mathrm{ord}_{u_l}(f)}}.$$

The second step is the map $\beta : \big(\mathbb{C}[Z]\big)[W_1, \ldots, W_n] \to S(V)[W_1, \ldots, W_n]$ sending $T_k \mapsto 1$ for each $1 \le k \le m$. Finally, we define the map $h : \mathbb{C}[Z] \to S(V)[W_1, \ldots, W_n]$ by composing the two steps, i.e. $h := \beta \circ \alpha$.

Theorem 2.19. *We have*

$$\mathcal{R}(Y) \cong S(V)[W_1, \ldots, W_n] / \big(f^h ; f \in \mathfrak{p} \big),$$

with $\mathrm{Cl}(Y)$*-grading given by* $\deg(S_{ij}) = [D_i]$ *and* $\deg(W_l) = [Y_l]$.

Proof. As $\Gamma(X, \mathcal{O}_X^*) = \mathbb{C}^*$, we have the canonical quotient construction $\pi : \widetilde{X} \to X$, where π is a good geometric quotient (cf. [CLS11, Theorem 5.1.11]). We set $\widetilde{Y} := \pi^{-1}(Y)$. By [ADHL10, Construction 5.1.4], the scheme-theoretic fiber $\widetilde{X} \times_X Y$ is reduced. There is a canonical toric closed embedding $\phi : \widetilde{X} \hookrightarrow \widehat{X}$ such that $\pi = p \circ \phi$.

Note that ϕ^* sends $f \mapsto f^\beta$. The assertion now follows from Proposition 2.18 and [Hau08, Theorem 2.6] if we show that the ideal $\widehat{\mathfrak{p}}$ of $\widehat{Y}$ in $\widehat{X}$ is $\mathfrak{p}^\alpha := (f^\alpha; f \in \mathfrak{p})$.

We use the map ψ from Remark 2.10. We have $\widehat{\mathfrak{p}} = \mathbb{I}(\Gamma \cdot \psi(\mathbb{V}(\mathfrak{p})))$. Every $f^\alpha \in \mathfrak{p}^\alpha$ vanishes on $\psi(\mathbb{V}(\mathfrak{p}))$ since $\psi^*(f^\alpha) = f \in \mathfrak{p}$, and f^α is a Γ-eigenvector, so $f^\alpha \in \widehat{\mathfrak{p}}$, and $\mathfrak{p}^\alpha \subseteq \widehat{\mathfrak{p}}$ follows. Now, let $g \in \widehat{\mathfrak{p}}$. As $\widehat{\mathfrak{p}}$ is a homogeneous ideal with respect to the $\mathfrak{X}(\Gamma)$-grading, all homogeneous components $g^{(\xi)}$ are in $\widehat{\mathfrak{p}}$. It is not difficult to see that $g^{(\xi)} = \big(\psi^*\big(g^{(\xi)}\big)\big)^\alpha \prod_{l=1}^n W_l^{d_l}$ for some exponents $d_l \in \mathbb{N}_0$. Since $\psi^*\big(g^{(\xi)}\big) \in \mathfrak{p}$, the inclusion $\widehat{\mathfrak{p}} \subseteq \mathfrak{p}^\alpha$ follows. □

Remark 2.20. If $\Gamma(G/H, \mathcal{O}^*_{G/H}) = \mathbb{C}^*$, we have $\widetilde{X} = \widehat{X}$, and ϕ is the identity.

It now only remains to show the last part of the Main Theorem.

Theorem 2.21. *If H is connected, $\mathcal{R}(Y)$ is a factorial ring.*

Proof. The finitely generated free abelian group $\Gamma(G/H, \mathcal{O}^*_{G/H})/\mathbb{C}^*$ is naturally isomorphic to the subgroup $\mathfrak{X}(C)^H \subseteq \mathfrak{X}(C)$ consisting of H-invariant characters. The quotient group $\mathfrak{X}(C)/\mathfrak{X}(C)^H$ is free as H is connected. Therefore there exists a decomposition $\mathfrak{X}(C) = \mathfrak{X}(C)^H \oplus \mathfrak{X}(C)^\dagger$, and we can choose the f_{ij} in such a way that C acts on the f_{ij} with characters belonging to $\mathfrak{X}(C)^\dagger$. It follows that for each $1 \le k \le m$ we have a one-parameter subgroup of C acting nontrivially only on the variable T_k. As $\widehat{Y}$ is C-invariant, we obtain $\widehat{Y} \cong \widetilde{Y} \times T$. Therefore the factoriality of $\mathcal{R}(Y) \cong \Gamma(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}})$ follows from the factoriality of $\Gamma(\widehat{Y}, \mathcal{O}_{\widehat{Y}})$ (cf. Proposition 2.13). □

Finally, we provide the proof of Theorem 1.1.

Proof of Theorem 1.1. We have to show $\mathcal{V} = \operatorname{trop}(G/H \cap \mathbb{T}) \cap \mathcal{N}_\mathbb{Q}$. Using [Tev07, Lemma 2.2], we get the inclusion from the left to the right using Proposition 2.17 if Y_l intersects the $\mathbb{T}$-orbit which is dense in X_l. This is the case, since otherwise it would follow from Remark 2.14 that the codimension of Y_l in Y is at least two. If the inclusion from the right to the left did not hold, Lemma 2.16 would yield a non-existing G-invariant valuation. □

3. Generalization to arbitrary spherical homogeneous spaces

We now consider the case where the spherical homogeneous space is allowed to have nontrivial divisor class group and to be not quasiaffine. Let $\boldsymbol{G}$ be a connected reductive group and $\boldsymbol{H} \subseteq \boldsymbol{G}$ a spherical subgroup. We may again assume that $\boldsymbol{G}$ is of simply connected type, i.e. $\boldsymbol{G} = \boldsymbol{G}^{ss} \times \boldsymbol{C}$ where $\boldsymbol{G}^{ss}$ is semisimple simply connected and $\boldsymbol{C}$ is a torus.

We fix a Borel subgroup $\boldsymbol{B} \subseteq \boldsymbol{G}$ such that the base point $1 \in \boldsymbol{G}/\boldsymbol{H}$ lies in the open $\boldsymbol{B}$-orbit and denote by $\boldsymbol{\mathcal{D}} := \{\boldsymbol{D}_1, \dots, \boldsymbol{D}_r\}$ the set of $\boldsymbol{B}$-invariant prime divisors in $\boldsymbol{G}/\boldsymbol{H}$. For each $\boldsymbol{D}_i$ the pullback under the quotient map $\boldsymbol{G} \to \boldsymbol{G}/\boldsymbol{H}$ is a divisor with equation $\boldsymbol{f}_i \in \mathbb{C}[\boldsymbol{G}]$ where $\boldsymbol{f}_i$ is uniquely determined by being $\boldsymbol{C}$-invariant under left multiplication and $\boldsymbol{f}_i(1) = 1$ (cf. [Bri07, 4.1]). The group $\boldsymbol{H}$ acts from the right on each $\boldsymbol{f}_i$ with a character $\boldsymbol{\chi}_i \in \mathfrak{X}(\boldsymbol{H})$.

We define

$$G := \boldsymbol{G} \times (\mathbb{C}^*)^{\boldsymbol{\mathcal{D}}}$$
$$H := \{(h, \boldsymbol{\chi}_1(h), \dots, \boldsymbol{\chi}_r(h)); h \in \boldsymbol{H}\} \subseteq G,$$

and set $B := \boldsymbol{B} \times (\mathbb{C}^*)^{\boldsymbol{\mathcal{D}}}$. We have a quotient map $\boldsymbol{\pi} : G/H \to \boldsymbol{G}/\boldsymbol{H}$, which is a good geometric quotient by the torus $(\mathbb{C}^*)^{\boldsymbol{\mathcal{D}}}$. There is a natural isomorphism $H \cong \boldsymbol{H}$, and G/H is a spherical homogeneous space. The pullbacks of the $\boldsymbol{B}$-invariant

prime divisors in $\boldsymbol{G}/\boldsymbol{H}$ under the quotient map $\boldsymbol{\pi}$ are exactly the B-invariant prime divisors $D_1, \ldots, D_r$ in G/H.

We will soon see that the spherical homogeneous space G/H satisfies the hypotheses of Section 2. We therefore re-use the notation of Section 2. Where applicable, we use the same notation for the spherical homogeneous space $\boldsymbol{G}/\boldsymbol{H}$ as well, but in boldface symbols. In particular, $\boldsymbol{\mathcal{M}}$ is the weight lattice of $\boldsymbol{B}$-eigenvectors in the function field $\mathbb{C}(\boldsymbol{G}/\boldsymbol{H})$, $\boldsymbol{\mathcal{N}} = \operatorname{Hom}(\boldsymbol{\mathcal{M}}, \mathbb{Z})$ the dual lattice, $\boldsymbol{\mathcal{N}}_{\mathbb{Q}} = \boldsymbol{\mathcal{N}} \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\boldsymbol{\mathcal{V}}$ is the valuation cone of $\boldsymbol{G}/\boldsymbol{H}$.

We consider a spherical embedding of $\boldsymbol{G}/\boldsymbol{H} \hookrightarrow \boldsymbol{Y}$ which contains only non-open $\boldsymbol{G}$-orbits of codimension one given by a fan $\boldsymbol{\Sigma}$ in $\boldsymbol{\mathcal{N}}_{\mathbb{Q}}$ and assume $\Gamma(\boldsymbol{Y}, \mathcal{O}^*_{\boldsymbol{Y}}) = \mathbb{C}^*$. We denote by $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n$ the $\boldsymbol{G}$-invariant prime divisors in $\boldsymbol{Y}$.

Before explaining the details, we give an overview over how we are going to obtain a description of the Cox ring $\mathcal{R}(\boldsymbol{Y})$ using the results of Section 2 and the spherical homogeneous space G/H. Recall the decomposition $\mathcal{M} = \mathcal{M}_V \oplus \mathcal{M}_T$. We have a corresponding decomposition of the dual lattice $\mathcal{N} = \mathcal{N}_V \oplus \mathcal{N}_T$. These decompositions depend on the choice of the f_i. We will show that for a suitable choice there is an isomorphism $(\mathcal{N}_T)_{\mathbb{Q}} \cong \boldsymbol{\mathcal{N}}_{\mathbb{Q}}$ which preserves the respective valuation cones. Via this isomorphism we obtain a fan Σ in $\mathcal{N}_{\mathbb{Q}}$ from the fan $\boldsymbol{\Sigma}$ in $\boldsymbol{\mathcal{N}}_{\mathbb{Q}}$ as well as a corresponding spherical embedding $G/H \hookrightarrow Y$. Then we construct the ring $\mathcal{R}(Y)$ exactly as in Section 2. Note that we may have $\Gamma(Y, \mathcal{O}^*_Y) \neq \mathbb{C}^*$, so the ring $\mathcal{R}(Y)$ may not be the Cox ring of Y. This ring will, however, be the Cox ring $\mathcal{R}(\boldsymbol{Y})$ of $\boldsymbol{Y}$.

Proposition 3.1. *We have the following commutative diagram, where the top row is an exact sequence.*

$$\begin{tikzcd}
0 \arrow[r] & \boldsymbol{\mathcal{M}} \arrow[r] & \mathfrak{X}(\boldsymbol{C}) \oplus \mathfrak{F}(\boldsymbol{\mathcal{D}}) \arrow[r] \arrow[d, "\cong"'] & \operatorname{Pic}_{\boldsymbol{G}}(\boldsymbol{G}/\boldsymbol{H}) \arrow[r] & 0 \\
& & \mathfrak{X}(G) \arrow[r, "\text{res}"'] & \mathfrak{X}(H) \arrow[u, "\cong"'] &
\end{tikzcd}$$

Proof. We denote by $\mathfrak{F}(\boldsymbol{\mathcal{D}})$ the free abelian group with basis $\boldsymbol{\mathcal{D}}$, res is the restriction map, and the map $\mathfrak{X}(\boldsymbol{C}) \oplus \mathfrak{F}(\boldsymbol{\mathcal{D}}) \to \operatorname{Pic}_{\boldsymbol{G}}(\boldsymbol{G}/\boldsymbol{H})$ is as in [Bri07, Proposition 4.1.1]. The left-hand isomorphism is obvious, and the right-hand isomorphism is the standard construction (cf. [KKLV89, 2.1] or [Tim11, after Proposition 2.4]) preceded by inversion on $\mathfrak{X}(H)$. It is not difficult to see that the diagram commutes. We denote by $\rho(\boldsymbol{D}_i) \in \boldsymbol{\mathcal{N}}_{\mathbb{Q}}$ the vector corresponding to the restriction of the discrete valuation induced by $\boldsymbol{D}_i \in \boldsymbol{\mathcal{D}}$ and set $d_i(\mu) := (\rho(\boldsymbol{D_i}))(\mu) \in \mathbb{Z}$. We define the map $\boldsymbol{\mathcal{M}} \to \mathfrak{X}(\boldsymbol{C}) \oplus \mathfrak{F}(\boldsymbol{\mathcal{D}})$ by

$$\mu \mapsto \sum_{i=1}^{r} d_i(\mu) \boldsymbol{D_i} - \mu|_{\boldsymbol{C}}.$$

If $d_i(\mu) = 0$ for all $1 \leq i \leq r$ and μ is the $\boldsymbol{B}$-weight of $f \in \mathbb{C}(\boldsymbol{G}/\boldsymbol{H})^{(\boldsymbol{B})}$, it follows that $\operatorname{div}(f) = 0$, so f is a unit in $\Gamma(\boldsymbol{G}/\boldsymbol{H}, \mathcal{O}_{\boldsymbol{G}/\boldsymbol{H}})$. This means we have $f \in \mathfrak{X}(\boldsymbol{C})$, so $\mu|_{\boldsymbol{C}} = 0$ if and only if $\mu = 0$. Therefore the map is injective. Using [Bri07, Proposition 4.1.1], we obtain the exactness of the top row. □

Corollary 3.2. *The spherical homogeneous space G/H is quasiaffine with trivial divisor class group.*

Proof. This follows from Proposition 3.1 using [Tim11, Theorem 2.5] and [Tim11, Theorem 3.12]. □

Remember that we use the notation from Section 2 freely. The quotient map $\boldsymbol{\pi}$ induces an inclusion $\boldsymbol{\pi}^* : \boldsymbol{\mathcal{M}} \hookrightarrow \mathcal{M}$, which is the restriction of the natural inclusion $\boldsymbol{\pi}^+ : \mathfrak{X}(\boldsymbol{B}) \to \mathfrak{X}(B)$. We also obtain the surjective dual map $\boldsymbol{\pi}_* : \mathcal{N}_{\mathbb{Q}} \to \boldsymbol{\mathcal{N}}_{\mathbb{Q}}$.

Proposition 3.3. *We have*

$$\mathcal{V} = \boldsymbol{\pi}_*^{-1}(\boldsymbol{\mathcal{V}}).$$

Proof. Let $v \in \mathcal{N}_{\mathbb{Q}}$ with $\boldsymbol{\pi}_*(v) = 0$. Interpreting v as map $v : \mathcal{M} \to \mathbb{Q}$, this means $v \circ \boldsymbol{\pi}^* = 0$. Therefore there exists an extension $v^+ : \mathfrak{X}(B) \to \mathbb{Q}$ of v with $v^+ \circ \boldsymbol{\pi}^+ = 0$. By [Kno91, Corollary 5.3], we obtain $v \in \mathcal{V}$, therefore $\boldsymbol{\pi}_*^{-1}(0) \subseteq \mathcal{V}$. Finally, we use [Kno91, Corollary 1.5]. $\square$

We now fix a suitable choice for the prime elements $f_1, \ldots, f_r \in \Gamma(G/H, \mathcal{O}_{G/H})$. Identifying the character lattice of $(\mathbb{C}^*)^{\boldsymbol{\mathcal{D}}}$ with $\mathfrak{F}(\boldsymbol{\mathcal{D}})$ as in Proposition 3.1, we have a natural inclusion $\mathfrak{F}(\boldsymbol{\mathcal{D}}) \subseteq \mathbb{C}[G]^*$. Then $f_i := \boldsymbol{f}_i \boldsymbol{D}_i^{-1} \in \mathbb{C}[G]$ is H-invariant under the action from the right, and f_i is a prime element in $\Gamma(G/H, \mathcal{O}_{G/H})$ with $\mathbb{V}(f_i) = D_i$. The torus $(\mathbb{C}^*)^{\boldsymbol{\mathcal{D}}}$ acts on $f_i \in \Gamma(G/H, \mathcal{O}_{G/H})$ with the character $\boldsymbol{D}_i$. As the B-weights of the kernel of the map $\operatorname{res} : \mathfrak{X}(G) \to \mathfrak{X}(H)$ are exactly the lattice $\mathcal{M}_T$, Proposition 3.1 yields an isomorphism $\gamma : \boldsymbol{\mathcal{M}} \to \mathcal{M}_T$.

Proposition 3.4. *The restricted map*

$$\boldsymbol{\pi}_*|_{(\mathcal{N}_T)_{\mathbb{Q}}} : (\mathcal{N}_T)_{\mathbb{Q}} \to \boldsymbol{\mathcal{N}}_{\mathbb{Q}}$$

is dual to the isomorphism $\gamma : \boldsymbol{\mathcal{M}} \to \mathcal{M}_T$. In particular, it is itself an isomorphism.

Proof. We have to show that for each $v \in \mathcal{N}_{\mathbb{Q}}$ with $v|_{\mathcal{M}_V} = 0$ and each $\mu \in \boldsymbol{\mathcal{M}}$ we have $v(\gamma(\mu)) = v(\boldsymbol{\pi}^*(\mu))$. It therefore suffices to show that $\gamma(\mu) - \boldsymbol{\pi}^*(\mu) \in \mathcal{M}_V$ for each $\mu \in \boldsymbol{\mathcal{M}}$.

Let $\mu \in \boldsymbol{\mathcal{M}}$ be the $\boldsymbol{B}$-weight of the $\boldsymbol{B}$-eigenvector $f \in \mathbb{C}(\boldsymbol{G}/\boldsymbol{H}) \subseteq \mathbb{C}(\boldsymbol{G})$. Using the notation from the proof of Proposition 3.1, we necessarily have

$$f = c \prod_{i=1}^{r} \boldsymbol{f}_i^{d_i(\mu)},$$

where $c \in \mathbb{C}[\boldsymbol{G}]^*$ has left $\boldsymbol{B}$-weight $\mu|_{\boldsymbol{C}}$. Therefore we obtain

$$\gamma(\mu) = -\sum_{i=1}^{r} d_i(\mu) \boldsymbol{D}_i + \mu|_{\boldsymbol{C}} \qquad \text{and} \qquad \boldsymbol{\pi}^*(\mu) = \mu|_{\boldsymbol{C}} + \sum_{i=1}^{r} d_i(\mu) \chi_{\boldsymbol{f}_i},$$

where $\chi_{\boldsymbol{f}_i}$ is the left $\boldsymbol{B}$-weight of $\boldsymbol{f}_i$, hence $\gamma(\mu) - \boldsymbol{\pi}^*(\mu) \in \mathcal{M}_V$. $\square$

Via $\boldsymbol{\pi}_*|_{(\mathcal{N}_T)_{\mathbb{Q}}}$ we obtain the fan Σ in $\mathcal{N}_{\mathbb{Q}}$ with associated spherical embedding $G/H \hookrightarrow Y$ from the fan $\boldsymbol{\Sigma}$ in $\boldsymbol{\mathcal{N}}_{\mathbb{Q}}$.

Remark 3.5. There is a good geometric quotient of the whole toric variety X by the action of $(\mathbb{C}^*)^{\boldsymbol{\mathcal{D}}}$. This means that $\boldsymbol{\pi}$ can be extended to a quotient $\boldsymbol{\pi} : X \to \boldsymbol{X}$. We obtain the following natural commutative diagram.

$$\begin{tikzcd}
(\mathcal{N}_T)_{\mathbb{Q}} \arrow[r, hook] & \mathcal{N}_{\mathbb{Q}} \arrow[r, "\boldsymbol{\pi}_*"] \arrow[d, hook] & \boldsymbol{\mathcal{N}}_{\mathbb{Q}} \arrow[d, hook] \\
& N_{\mathbb{Q}} \arrow[r, "\boldsymbol{\pi}_*"] & \boldsymbol{N}_{\mathbb{Q}}
\end{tikzcd}$$

Similarly to Section 2, we obtain $\boldsymbol{Y}$ as closure of $\boldsymbol{G}/\boldsymbol{H}$ inside $\boldsymbol{X}$, and the embedding $\boldsymbol{Y} \hookrightarrow \boldsymbol{X}$ is neat.

Theorem 3.6. *We have*

$$\mathcal{R}(\boldsymbol{Y}) \cong S(V)[W_1, \dots, W_n]/\big(f^h; f \in \mathfrak{p}\big),$$

with $\mathrm{Cl}(\boldsymbol{Y})$*-grading given by* $\deg(S_{ij}) = [\boldsymbol{D}_i]$ *and* $\deg(W_l) = [\boldsymbol{Y}_l]$.

Proof. As in Section 2, we construct the good quotient $p : \widehat{X} \to X$, and we have the canonical quotient construction $\pi : \widetilde{X} \to \boldsymbol{X}$ as well as a canonical toric closed embedding $\phi : \widetilde{X} \to \widehat{X}$ such that $\pi = \boldsymbol{\pi} \circ p \circ \phi$. The result now follows as in Section 2. $\square$

Theorem 3.7. *If* $\boldsymbol{H}$ *is connected,* $\mathcal{R}(\boldsymbol{Y})$ *is a factorial ring.*

Proof. We embed Y into another toric variety X' using primes $f_i' \in \Gamma(G/H, \mathcal{O}_{G/H})$ satisfying the requirements of the proof of Theorem 2.21 instead of the primes f_i. The f_i' can be chosen in such a way that for each $1 \le i \le r$ there is a unit c_i such that $f_i' = c_i f_i$ and c_i is the product of elements of $\{g_k\}_{k=1}^m$. As a basis for the G-module $\langle G \cdot f_i' \rangle$ we then choose $\{c_i f_{ij}\}_{j=1}^{s_i}$. In this case, there is a toric isomorphism $X' \cong X$ which fixes G/H and therefore Y as well. We obtain the following commutative diagram.

$$\begin{tikzcd} & \widehat{X}' \arrow[r, "p'"] & X' \arrow[d, "\cong"] & \\ \widetilde{X} \arrow[ur, "\phi'"] \arrow[r, "\phi"'] & \widehat{X} \arrow[r, "p"'] & X \arrow[r, "\boldsymbol{\pi}"'] & \boldsymbol{X} \end{tikzcd}$$

The factoriality of $\mathcal{R}(\boldsymbol{Y})$ now follows as in Theorem 2.21. $\square$

Theorem 3.8. *If* $\boldsymbol{G}/\boldsymbol{H}$ *is horospherical, we have*

$$\mathcal{R}(\boldsymbol{Y}) \cong \mathcal{R}(\boldsymbol{G}/\boldsymbol{P})[W_1, \dots W_n],$$

where $\boldsymbol{P} := N_{\boldsymbol{G}}(\boldsymbol{H})$.

Proof. As $\boldsymbol{G}/\boldsymbol{H}$ being horospherical implies that G/H is horospherical as well, the ideal $\mathfrak{p}$ is $\mathcal{M}$-graded by Proposition 2.6. This means, for each $f \in \mathfrak{p}$ and $\mu \in \mathcal{M}$ we have $f^{(\mu)} \in \mathfrak{p}$. As $\big(f^{(\mu)}\big)^\alpha = f^{(\mu)}$, the ideal $(f^h; f \in \mathfrak{p})$ can be generated by elements which do not contain any of the variables W_l. It follows that $\mathcal{R}(\boldsymbol{Y}) \cong R[W_1, \dots, W_n]$ for some ring R which depends only on the homogeneous space.

It only remains to show that $R \cong \mathcal{R}(\boldsymbol{G}/\boldsymbol{P})$. We have canonical maps

$$\boldsymbol{G} \longrightarrow \boldsymbol{G}/\boldsymbol{H} \xrightarrow{\cong} (\boldsymbol{G} \times \boldsymbol{T})/\boldsymbol{P} \longrightarrow \boldsymbol{G}/\boldsymbol{P},$$

where $\boldsymbol{T} := \boldsymbol{P}/\boldsymbol{H}$ is a torus, $\boldsymbol{P}$ acts on $\boldsymbol{T}$ via $\boldsymbol{P} \to \boldsymbol{T}$, and $\boldsymbol{P}$ acts on $\boldsymbol{G} \times \boldsymbol{T}$ via $p \cdot (g, t) := (gp^{-1}, pt)$. The last map has fibers isomorphic to $\boldsymbol{T}$ and is a trivial fibration over the open orbit of any Borel subgroup of $\boldsymbol{G}$ (cf. [BM12, after Theorem 2.3]). The pullbacks of the $\boldsymbol{B}$-invariant prime divisors in $\boldsymbol{G}/\boldsymbol{P}$ are exactly the $\boldsymbol{B}$-invariant prime divisors in $\boldsymbol{G}/\boldsymbol{H}$. In particular, $\boldsymbol{P}$ acts from the right on each $\boldsymbol{f}_i \in \mathbb{C}[\boldsymbol{G}]$ with an extension of the character $\boldsymbol{\chi}_i \in \mathfrak{X}(\boldsymbol{H})$ which we will also call $\boldsymbol{\chi}_i \in \mathfrak{X}(\boldsymbol{P})$. We define

$$P := \{(p, \boldsymbol{\chi}_1(p), \dots, \boldsymbol{\chi}_r(p)); p \in \boldsymbol{P}\} \subseteq G.$$

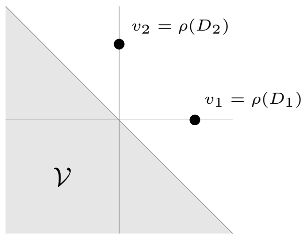
It is not difficult to see that G/H is isomorphic to $(G \times \boldsymbol{T})/P$ where $P \cong \boldsymbol{P}$ acts on $G \times \boldsymbol{T}$ via $p \cdot (g, t) := (gp^{-1}, pt)$. As all characters of P can be extended to G (cf. Proposition 3.1), we obtain $G/H \cong G/P \times \boldsymbol{T}$, and the result follows. $\square$

4. Examples

Example 4.1. Consider $G := \mathrm{SL}(3)$ and $H := U$, the set of unipotent upper triangular matrices. Let $B \subseteq G$ be the Borel subgroup of upper triangular matrices, and let G act linearly on $\mathbb{C}^3 \times \mathbb{C}^3$ by acting naturally on the first factor and with the contragredient action on the second factor. We denote the coordinates of the first factor by S_{11}, S_{12}, S_{13} and the coordinates of the second factor by S_{21}, S_{22}, S_{23}. Then the point $((1,0,0),(0,0,1))$ has isotropy group U, and its orbit is $\mathbb{V}(S_{11}S_{21} + S_{12}S_{22} + S_{13}S_{23})$. The homogeneous space G/H is horospherical with B-invariant prime divisors $D_1 := \mathbb{V}(S_{13})$ and $D_2 := \mathbb{V}(S_{21})$ as well as quasiaffine with trivial divisor class group. Consider the embedding Y corresponding to a fan containing exactly n one-dimensional cones. By Theorem 3.8 and setting $P := N_G(H)$, we obtain

$$\mathcal{R}(G/P) \cong \mathbb{C}[S_{11}, S_{12}, S_{13}, S_{21}, S_{22}, S_{23}]/(S_{11}S_{21} + S_{12}S_{22} + S_{13}S_{23})$$
$$\mathcal{R}(Y) \cong \mathcal{R}(G/P)[W_1, \dots, W_n].$$

Example 4.2. Consider $G := \mathrm{SL}(d)$ for $d \geq 3$ and $H := \mathrm{SL}(d-1)$ embedded as the lower-right entries of $\mathrm{SL}(d)$. The case $d = 3$ has been studied in [Pau83] and [Pau89]. Let $B \subseteq G$ be the Borel subgroup of upper triangular matrices, and let G act linearly on $\mathbb{C}^n \times \mathbb{C}^n$ by acting naturally on the first factor and with the contragredient action on the second factor. We denote the coordinates of the first factor by $S_{11}, \dots, S_{1d}$ and the coordinates of the second factor by $S_{21}, \dots, S_{2d}$. Then the point $((1,0,\dots,0),(1,0,\dots,0))$ has isotropy group $\mathrm{SL}(d-1)$, and its orbit is $\mathbb{V}\big(\sum_{j=1}^{d} S_{1j}S_{2j} - 1\big)$. The homogeneous space G/H is spherical with B-invariant prime divisors $D_1 := \mathbb{V}(S_{1d})$ and $D_2 := \mathbb{V}(S_{21})$ as well as affine with factorial coordinate ring. By Theorem 1.1, we obtain $\mathcal{V} = \{v_1^* + v_2^* \leq 0\}$. Consider the embedding Y corresponding to the fan containing exactly the one-dimensional cones having primitive lattice generators $(p_1, q_1), \dots, (p_n, q_n) \in \mathcal{N}$ with respect to the basis $\{v_1, v_2\}$. The following picture illustrates $\mathcal{N}_{\mathbb{Q}}$.



By Theorem 2.19, we obtain

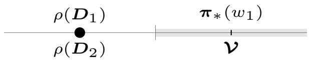
$$\mathcal{R}(Y) \cong \mathbb{C}[S_{1j}, S_{2j}, W_1, \dots, W_n]_{j=1}^{d} \Bigg/ \left(\sum_{j=1}^{d} S_{1j}S_{2j} - W_1^{-p_1-q_1} \cdots W_n^{-p_n-q_n} \right).$$

Example 4.3. Consider $\boldsymbol{G} := \mathrm{SL}(2)$ and $\boldsymbol{H} := T$, the diagonal torus. Let $\boldsymbol{B} \subseteq \boldsymbol{G}$ be the Borel subgroup of upper triangular matrices. Then, considering the orbit of $([1:0],[0:1])$ in $\mathbb{P}^1 \times \mathbb{P}^1$, we obtain $\boldsymbol{G}/\boldsymbol{H} \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathrm{diag}\big(\mathbb{P}^1\big)$, and there are two $\boldsymbol{B}$-invariant prime divisors. From

$$\mathbb{C}[\boldsymbol{G}] = \mathbb{C}[M_{11}, M_{12}, M_{21}, M_{22}]/(M_{11}M_{22} - M_{12}M_{21} - 1)$$

we obtain $\boldsymbol{f}_1 = M_{21}$ and $\boldsymbol{f}_2 = M_{22}$. We can now construct $G = \boldsymbol{G} \times (\mathbb{C}^*)^{\boldsymbol{\mathcal{D}}}$ and H. It is not difficult to see that G/H is the orbit of the point $((1,0),(0,1),1)$ in $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^*$, where $\boldsymbol{G}$ acts naturally on both factors $\mathbb{C}^2$ and trivially on $\mathbb{C}^*$ while $(\mathbb{C}^*)^{\boldsymbol{\mathcal{D}}}$ acts with the characters $-\boldsymbol{D}_1$ and $-\boldsymbol{D}_2$ respectively on the factors $\mathbb{C}^2$ as well as with $-\boldsymbol{D}_1 - \boldsymbol{D}_2$ on $\mathbb{C}^*$. Denoting by S_{11}, S_{12} and S_{21}, S_{22} the

coordinates of the factors $\mathbb{C}^2$ and by T_1 the coordinate of the factor $\mathbb{C}^*$, we obtain $G/H = \mathbb{V}(S_{11}S_{22} - S_{12}S_{21} - T_1)$. By Theorem 1.1, we obtain $\mathcal{V} = \{v_1^* + v_2^* \leq w_1^*\}$. Using Proposition 3.4 and Proposition 3.1, we see that $\boldsymbol{\pi}_*(w_1)$ is a basis of $\boldsymbol{\mathcal{N}}_\mathbb{Q}$ and that $\boldsymbol{\pi}_*(v_1) = \boldsymbol{\pi}_*(v_2) = -\boldsymbol{\pi}_*(w_1)$. By Proposition 3.3, $\boldsymbol{\mathcal{V}} = \{\boldsymbol{\pi}_*(w_1)^* \geq 0\}$ follows. The following picture illustrates $\boldsymbol{\mathcal{N}}_\mathbb{Q}$.



By Theorem 3.6, we have

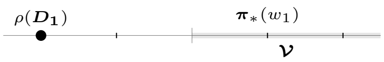
$$\mathcal{R}(\boldsymbol{G}/\boldsymbol{H}) \cong \mathbb{C}[S_{11}, S_{12}, S_{21}, S_{22}]/(S_{11}S_{22} - S_{12}S_{21} - 1).$$

The only nontrivial embedding of $\boldsymbol{G}/\boldsymbol{H}$ is the embedding $\boldsymbol{Y} := \mathbb{P}^1 \times \mathbb{P}^1$ given by the cone $\mathbb{Q}_{\geq 0}\boldsymbol{\pi}_*(w_1)$. In this case, Theorem 3.6 yields

$$\mathcal{R}(\boldsymbol{Y}) \cong \mathbb{C}[S_{11}, S_{12}, S_{22}, S_{21}, W_1]/(S_{11}S_{22} - S_{12}S_{21} - W_1),$$

which is isomorphic to the polynomial ring in four variables.

Example 4.4. Consider $\boldsymbol{G} := \mathrm{SL}(2)$ and $\boldsymbol{H} := N_{\boldsymbol{G}}(T)$ where T is the diagonal torus. Let $\boldsymbol{B} \subseteq \boldsymbol{G}$ be the Borel subgroup of upper triangular matrices. The group $\boldsymbol{H}$ consists of two connected components, the identity component $\boldsymbol{H}^\circ = \left\{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right); \lambda \in \mathbb{C}^*\right\}$ and a second component $\boldsymbol{H}^\dagger = \left\{\left(\begin{smallmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{smallmatrix}\right); \lambda \in \mathbb{C}^*\right\}$. There exists a character $\boldsymbol{\chi} \in \mathfrak{X}(\boldsymbol{H})$ with $\boldsymbol{\chi}|_{\boldsymbol{H}^\circ} = 1$ and $\boldsymbol{\chi}|_{\boldsymbol{H}^\dagger} = -1$. Consider the complex vector space $\mathrm{Sym}(2 \times 2) = \left\{\left(\begin{smallmatrix} s_{12} & s_{13} \\ s_{13} & s_{11} \end{smallmatrix}\right); s_{ij} \in \mathbb{C}^*\right\}$ of symmetrical 2 by 2 matrices. Let $\boldsymbol{G} \times \mathbb{C}^*$ act on $\mathrm{Sym}(2 \times 2) \times \mathbb{C}^*$ via $(g, t) \cdot (x, y) := (t^{-1}gxg^T, t^{-2}y)$. The isotropy group of the point $\left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), 1\right)$ is $H' := \{(h, \boldsymbol{\chi}(h)), h \in \boldsymbol{H}\}$ and its orbit is the closed subset $\mathbb{V}(S_{11}S_{12} - S_{13}^2 - T_1)$. We observe that $(\boldsymbol{G} \times \mathbb{C}^*)/H'$ is a spherical homogeneous space with one $\boldsymbol{B} \times \mathbb{C}^*$-invariant prime divisor $D_1 := \mathbb{V}(S_{11})$ and that it coincides with G/H as defined in Section 3. By Theorem 1.1, we obtain $\mathcal{V} = \{2v_1^* \leq w_1^*\}$. Using Proposition 3.4 and Proposition 3.1, we see that $\boldsymbol{\pi}_*(w_1)$ is a basis of $\boldsymbol{\mathcal{N}}_\mathbb{Q}$ and that $\boldsymbol{\pi}_*(v_1) = -2\boldsymbol{\pi}_*(w_1)$. By Proposition 3.3, $\boldsymbol{\mathcal{V}} = \{\boldsymbol{\pi}_*(w_1)^* \geq 0\}$ follows. The following picture illustrates $\boldsymbol{\mathcal{N}}_\mathbb{Q}$.



Consider the trivial embedding $\boldsymbol{Y} := \boldsymbol{G}/\boldsymbol{H}$. We have $\mathrm{Cl}(\boldsymbol{Y}) \cong \mathbb{Z}/2\mathbb{Z}$ with generator $[\boldsymbol{D}_1] \in \mathrm{Cl}(\boldsymbol{Y})$. By Theorem 3.6, we obtain

$$\mathcal{R}(\boldsymbol{Y}) \cong \mathbb{C}[S_{11}, S_{12}, S_{13}]/(S_{11}S_{12} - S_{13}^2 - 1),$$

which is a non-factorial ring.

Acknowledgments

I am highly grateful to my teacher, Victor Batyrev, for his guidance and support. I would also like to thank Jürgen Hausen and Johannes Hofscheier for some useful discussions.

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Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany

E-mail address: `giuliano.gagliardi@uni-tuebingen.de`